It's Not Just Notation: Valuing Children's Representations

Mathematics teachers often say to their students, "Show your work," when what they really mean is, "Show this to me as I showed it to you" or "Show *my* work." During our first few years as mathematics teachers, we spent much of our time trying to make our students think as we thought—to use the symbols that we were using in the ways in which we were using them. We find it uncomfortable to admit that during those years we spent little time trying to find out what our students were thinking. We valued the process and product looking like ours more than the process and product being theirs. We expected our students to use conventional representations before we had given them sufficient time to develop mathematical concepts.

Jill A. Perry and Sandra L. Atkins

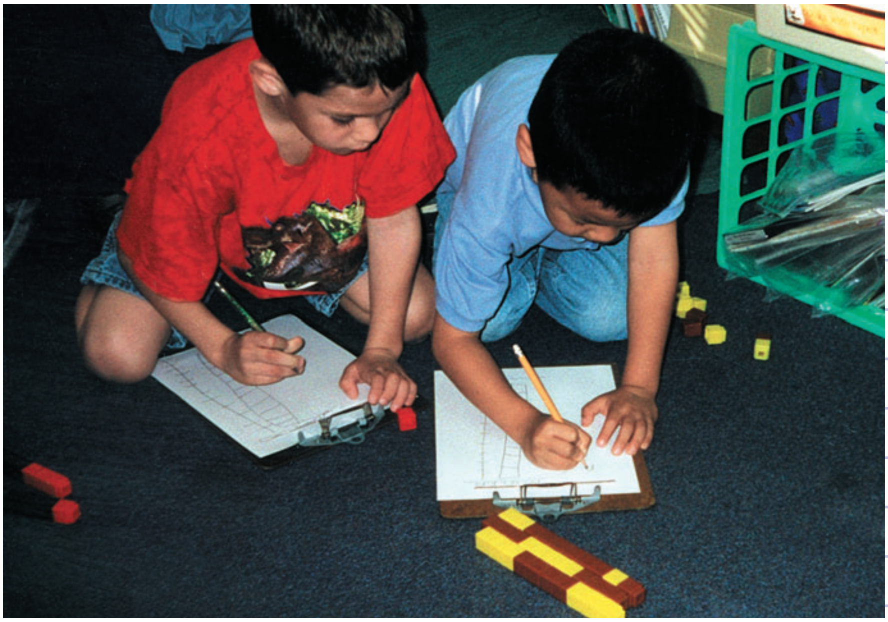
At different points in our careers and in different ways, we realized that there must be a balance between personal representations and conventional representations. This stance gradually became a part of our philosophies and pedagogies. We since have found that focusing on the meaning that students make of the mathematics they do and the representations they use is more revealing of their understanding than is looking for our representations in their work.

Jill Perry, perry@rowan.edu, teaches general and mathematics pedagogy courses to preservice teachers at Rowan University in Glassboro, New Jersey. She is interested in the connections among mathematics, language, literacy, representation, and communication. Sandra Atkins, sandra_atkins@mcgraw-hill.com, is a senior mathematics consultant for Wright Group/McGraw-Hill in Bothell, Washington. Her interests are in mathematical communication and the relationship of mathematical reasoning to representations.

A Notion of Representation

Many of us assume that children's representations are little more than their haphazard recording of mathematical work. However, in *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics (2000) suggests that "teachers can gain valuable insights into students' ways of interpreting and thinking about mathematics by looking at their representations" (p. 68). This view implies that children's representations provide clues to the ways that children make sense of the mathematics that they are learning. Holding this view of representations also helps us see representations as paths into dialogues with children about their mathematical thinking; it also helps us view these representations as "bridges . . . to more conventional ones, when appropriate" (NCTM 2000, p. 68).

The challenges lie in recognizing and developing tasks that create opportunities for children to represent their mathematical thinking in their own ways, as well as in knowing when to be quiet and when to intervene. This process requires trusting the children's abilities to make sense of mathematics and the teacher's ability to recognize the mathematical connections, develop related tasks, allow for exploration, and introduce and facilitate the use of conventional representations.


Photograph by Sandra L. Atkins; all rights reserved

Understanding Children's Understandings through Their Representations

In our work with children, we have learned a great deal about the importance of examining children's representations and using those representations as starting points for dialogues with children. The classroom episodes presented in this article illustrate some of what we can learn about children's mathematical understandings from their representations and our discussions with the children about them.

"Making tens" in first grade

Students in Mrs. Kalew's first-grade class worked in pairs as they explored "tenness" through an activity called "Making Tens." Each child had a small basketful of Unifix cubes, which were all the same color but a different color from his or her partner's cubes. For this activity (TERC Project 1998), the children were to take turns rolling a number cube and adding cubes to the tower based on the number that they rolled. Once the tower had a height of ten cubes, the children were to start a new tower, with play continuing until they had built three "towers of ten" together. When all three towers were built, the children were to record their towers for the class discussion. Mrs. Kalew did not give the children any indication of a preferred method for recording their towers; they could use "pictures, numbers, or words."

How the children represented their thinking

During the class discussion, the children sat in a circle on the floor and shared their representations by showing their papers or describing what they had recorded. Mrs. Kalew collected the representations on a whiteboard easel as the children shared them. Some of the children had drawn and colored pictures of the towers (see **fig. 1**), some had written arithmetic expressions or equations, and others had written numerals in a two-column format (see **fig. 2**). In most cases, the children recorded the towers in the same ways as their partners. All the students who drew pictures of their towers drew all three towers as they appeared, preserving the color relationships. Of those who used conventional symbolic notation, all recorded the towers as sums of two numbers, even though some towers looked like those in **figure 1.** For example, Linda recorded the towers in **figure 1** as 4 + 6, 6 + 4, and 5 + 5, not as 1 + 6 + 3, 2 + 4 + 4, and 1 + 2 + 4 + 3. Her partner, Violet, used a two-column format and recorded the towers as shown in **figure 2.** Amazed by the multitude of representations that the children created, we were anxious to talk to them about their representations.

What the children found important

After the children finished sharing their representations, Mrs. Kalew talked to them about two things she had noticed. First, she pointed out that some of the towers the children had drawn looked like the first one that Trevor and Alan had built (see **fig. 3**), with more than six cubes of the same color in a row. Despite having used a number cube with the digits 1 through 6 on it, the boys had eight contiguous blue cubes in their first tower. At first we thought that the boys had an intuitive understanding of the associative and commutative properties of addition and that they had put the numbers together by combining like cubes. After asking them about the tower, however, we learned that this was not the case.

When Mrs. Kalew asked how they got eight blue cubes in a row, Trevor told her that he and Alan added cubes only to their own "end" of the tower. For example, for the towers shown in **figure 3,** the first few turns for Trevor and Alan might have been as follows:

- Trevor rolls a 5 and puts five blue cubes together (see **fig. 4a**).
- Alan rolls a 2 and attaches two white cubes to the tower (see **fig. 4b**).

FIGURE 1

Three “towers of ten”

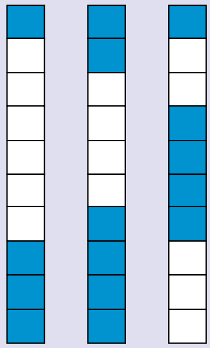


FIGURE 2

Violet’s representation of the towers in figure 1

Blue	White
4	6
6	4
5	5

FIGURE 3

Same-color tower building

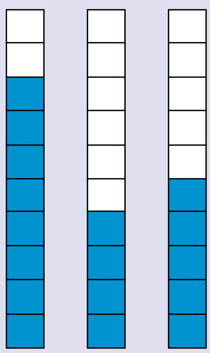
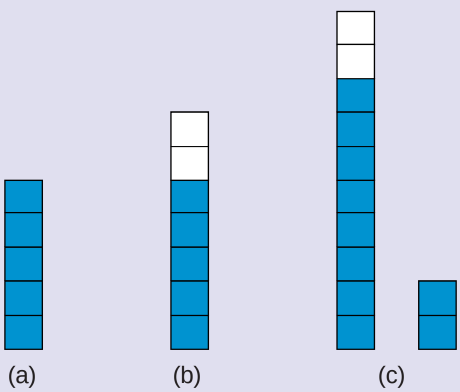


FIGURE 4

Step-by-step, same-color tower building



- Trevor rolls a 5, picks up five blue cubes, and attaches three of them to the blue end of the tower to complete it. He then starts a new tower with the remaining two cubes (see **fig. 4c**).

The boys then told Mrs. Kalew that they put the same colors together so that they could tell who was winning. Their interpretation of the task was that they were to record their scores and find a winner, not find combinations of numbers whose sum is 10. Making tens was not a problem for them, so they changed the collaborative game into a competition. Melissa then added, to our bewilderment, that she and Becky had put the same colors together too, but they did it because it looked prettier. Sometimes when we think that children are showing us deep mathematical understandings, they really are showing us something much different. For all four of these children, their representations showed not only their understanding of the mathematics but also a different interpretation of the task.

The second thing that Mrs. Kalew pointed out was that some of the drawings looked like those in **figure 1,** but none of the numeric representations showed that kind of relationship, that is, 1 + 6 + 3, 2 + 4 + 4, and 1 + 2 + 4 + 3. When she asked why the students thought this was true, Violet said that she tried but could not fit all the numbers into her chart (see **fig. 2**). Her choice of format limited her ability to record the numbers as she thought she should. While Violet wanted to record more than her form of representation would allow, other children said that they did not record the numbers that way because they did not think that the method of recording was important; to them, “it didn’t matter except for the colors.” These children had interpreted the task in terms of color rather than number.

Even though some of the children had not focused on “tenness” while playing the game, the multiple representations presented during the class discussion offered many experiences with and discussions about “tenness.” Because many of the children used different forms of representation during the second round of play, we believe that the discussion provided a forum in which the children were able to test their own and others’ ideas and then adopt, adapt, or reject those ideas. It also allowed us to talk to the children about their representations.

Finding Out What They Mean, Not What You Think They Mean

What follows are two more classroom episodes of children’s thinking that show what we learn when we take the time to talk to them about their mathematical ideas. Probing beyond the initial answer makes the conversations fruitful. As we shall see again and again, a correct answer to the initial question does not always reflect a conventional understanding of the mathematics behind it.

Fractional parts of continuous models in fourth grade

When we visited Mrs. Meham's fourth-grade classroom, we learned a fourth grader's understanding of fractional representations. The children were using fraction strips (Burns 1992) to construct and explore fractional units. The activity began with children making their own fraction strip kits, which required careful attention to the language used by the instructor. To make thirds, the children were asked to fold a green strip of paper into three equal pieces and then cut or tear them apart. Each child then compared his or her pieces with another child's to make certain that, because they had all started with congruent wholes, all the thirds were the same. Once they had made all their pieces, they had an opportunity to explore the relationships among the pieces and engaged in a discussion that eventually led to the conventional names for each of the pieces (for example, *one-half, one-third*). A portion of a fraction strip kit is shown in **figure 5.**

After they had taken time to use the conventional names and become familiar with the pieces, the students were asked to show two-thirds. This was the children's first experience with these ideas as a class, and they were free to make conjectures and take risks. Some of the students immediately placed 2 one-third pieces end to end. Others picked up some of their pieces, looked them over, ran tests, and then decided to lay 2 one-third pieces end to end (see **fig. 6a**) or 1 one-third and 2 one-sixth pieces end to end (see **fig. 6b**). Grace also pondered the pieces for a while, but she did not create the same configurations as most of her classmates. She laid 2 one-sixth pieces end to end on her desk and then placed the one-third piece directly above them (see **fig. 6c**). When she finished this, she looked around at her classmates' representations. Despite finding only one that agreed with hers, she did not change her mind. She was confident that her representation was correct.

From the front of the classroom, Sandy saw Grace's representation and assumed that Grace's meaning was connected to the other children's meaning. Sandy thought that Grace had stacked the pieces instead of laying them end to end. Jill, who was sitting next to Grace, asked her how she had chosen the pieces. Grace held up the 2 one-sixth pieces and told Jill that those were the two equal pieces that fit on top of the one-third piece. By talking to

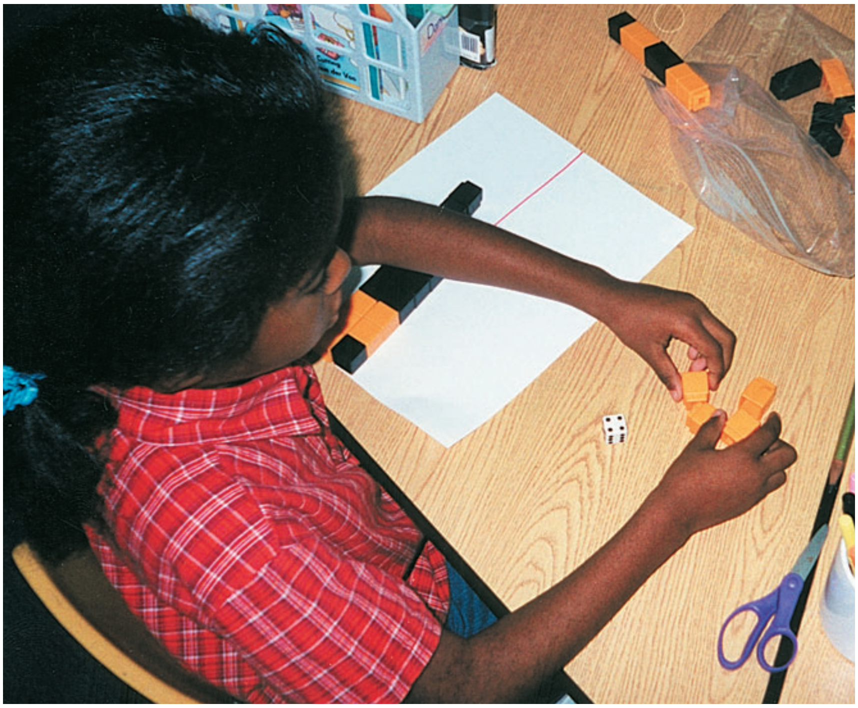


FIGURE 5

A portion of a fraction strip kit

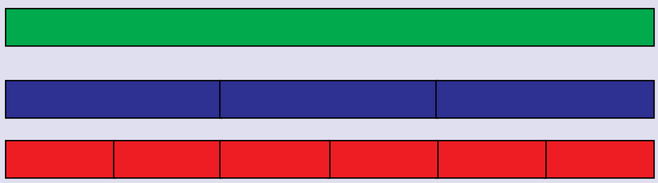
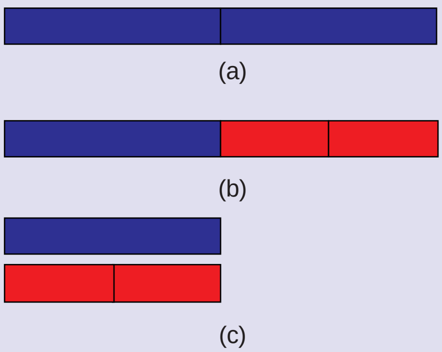


FIGURE 6

Students' attempts to show two-thirds



Grace, we learned that the conventional representation for two-thirds—that is, 2 of the one-third pieces—did not make sense to her. She saw two-thirds as "two equal pieces that make one-third." This illustrates the limitations of communication in a group of twenty or thirty children: it is difficult, if not impossible, to know what each child is thinking and to probe deeper. This puts educators in the dangerous position of imposing our meanings on their representations.

Fractional parts of groups of discrete objects in fourth grade

Another example of the fourth graders' understanding of fractional relationships illustrates the importance of exploring mathematical ideas in multiple contexts with multiple representations. Mrs. Meham and the children in her fourth-grade class were using bicolored disks to represent fractional relationships. Mrs. Meham asked the children to take out nine chips and show two-thirds yellow, one-third red. Each student showed one of the two diagrams in **figure 7** as a solution. Noticing that these were two different orientations of the same answer, we believed that all the children shared a common meaning for two-thirds, and one-third, of 9.

As a second task intended to provide the children with more experiences with fractional parts of groups of discrete objects, Mrs. Meham asked the children to lay out 15 disks, showing one-fifth yellow. They created the representations shown in **figure 8.** This time, it was clear that not all the children shared a common meaning for one-fifth of 15.

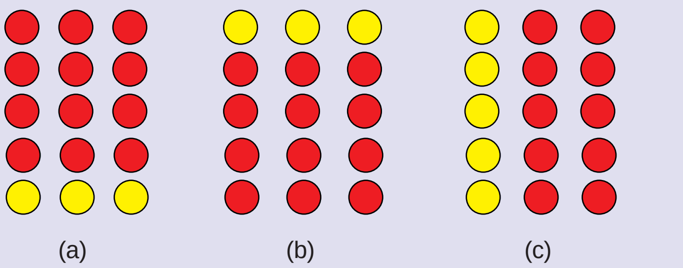
Mrs. Meham asked several of the children to explain their solutions by using transparent chips on the overhead projector. Students with the representations in **figure 8a** or **8b** explained that they had five groups of three and that the chips in one of those five groups should be yellow. The three students who provided the representation in **figure 8c** argued that there were five groups of three and that one chip in each of the five groups should be yellow. The two groups of children had the same meaning for the denominator but different meanings for the numerator. That is, all the children interpreted the denominator as indicating the number of equal groups of chips. However, children using the representations in **figure 8a** and **8b** saw the numerator conventionally, as the *number of groups* that have the given quality; in this case, all the chips in one of the five groups should be yellow. Children using the representation in **figure 8c** saw the numerator as the *number of chips* in each of the groups that have the given quality; in this case, one chip in each of the five groups should be yellow.

FIGURE 7

Students' representations of two-thirds and one-third

FIGURE 8

Varying student representations of one-fifth



Why was this difference not revealed in the first problem? Sometimes the problems that we pose unintentionally obscure our ability to understand our students' interpretations and uses of conventional notation. After much thought and discussion with colleagues and friends, we realized that the first problem masked the different conceptions of fraction notation because the number of chips used (9) was a square number. The explanations of "There are three groups of three and all the chips in one of those groups should be yellow" and "There are three groups of three and one chip in each of those groups should be yellow" both yield the same physical representation. This episode supports the notions that children should have multiple experiences with mathematical ideas and that teachers should take time to look at and talk to children about multiple representations.

What We Have Learned

Through working and talking with children, we have come to value mathematical representation as an integral part of children's mathematical understanding. We believe that we have a better chance of finding out what children understand by looking at and talking with them about their mathematical representations. From Mrs. Kalew's first graders, we learned that children's representations can give us insight into the children's mathematical thinking and also help us understand their interpretations of the tasks that we give them. Mrs. Meham's fourth graders helped us see that we must not impose our mathematical meanings on children's representations and that we must provide multiple experiences with mathematical concepts before we can expect children to understand and use conventional representations. From all the classroom episodes, we learned that just as dismissing children's representations leaves us with an incomplete picture of their mathematical understandings, so does examining their representations without eliciting further elaborations.

Conclusion

Ludwig Wittgenstein suggested that mathematical convention is a human invention (Bloor 1983). With this in mind, if we want children to play the same "game" by the same "rules," we must show them our mathematical conventions, for example, our ways of counting and meanings for operations. However, if our goal is for children to create and think critically about mathematics, then we first must give them time to work on mathematical tasks using their own representations. We cannot expect that children will always discover or invent conventional notation. We can expect that, given time to develop their own representations, children introduced to conventional notation will have the foundation necessary to make sense of experiences that they have with mathematical abstractions.

References

Bloor, David. *Wittgenstein: A Social Theory of Knowledge.* New York: Columbia University Press, 1983.

Burns, Marilyn. *About Teaching Mathematics: A K–8 Resource*. Sausalito, Calif.: Math Solutions Publications, 1992.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.

TERC Project. *Investigations in Number, Data, and Space (K–5)*. White Plains, N.Y.: Cuisenaire-Dale Seymour, 1998. ▲